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Lagrangian formulation of unsteady non-linear heat transfer problems

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SUMMARY
Lebon–Lambermont's variational principle is used to solve some problems of unsteady heat conduction in semi-infinite solids. These are characterized by temperature dependent heat conductivity and heat capacity. Two kinds of boundary conditions are applied: prescribed temperature history and prescribed heat flux across the surface. This latter problem is investigated by means of the technique proposed by Lardner and Rafalski–Zyszkowsky. The results are compared with exact ones, when available, or with those obtained from other variational principles like Biot's and Vujanovic's. In all cases, agreement is very satisfactory.

1. Introduction

In absence of heat sources, the equation of conduction of heat in an isotropic body is expressed, in cartesian coordinates, by

$$c\dot{T} = \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right), \tag{1.1}$$

c is the heat capacity per unit volume, k the heat conductivity, T the temperature while an upper dot stands for $\partial/\partial t$; the summation convention on repeated indices will be used throughout this paper. In most problems of practical interest, the thermal properties c and k do vary with the temperature so that (1.1) is a non-linear partial differential equation. Exact solutions may be obtained in some particular cases [1–3]. More frequently, variational methods or direct numerical methods, like the finite difference technique are used.

In the last years, there has been a considerable growth of interest in formulating variational principles. As is well known, dissipative phenomena cannot be described by exact variational criteria but rather by restricted principles [4]. These differ from the classical ones by the fact that some variables are frozen during the process of variation. Most of these criteria are formulated in the framework of thermodynamics of irreversible processes.

One of the best known is the principle of Glansdorff and Prigogine [5–7]. It states that a certain functional, the local potential, is extremum:

$$\delta\phi(\boldsymbol{v}, T, p, \ldots, \boldsymbol{v}^0, T^0, p^0, \ldots) = 0\,. \tag{1.2}$$

The local potential ϕ depends on two types of variables: the velocity $\boldsymbol{v}$, the temperature T, the pressure p, etc., which are submitted to variation and alias variables of the same kind $\boldsymbol{v}^0$, T^0, $p^0, \ldots$ which are kept constant during variation. However, in order to recover the correct conservation laws as Euler–Lagrange equations, the alias quantities are to be identified with the variables $\boldsymbol{v}$, T, $p, \ldots$ after the process of variation is completed. This criterion has been applied successfully to many problems of heat transfer and fluid mechanics [8–11].

Another approach, particularly suited in heat transfer and related problems like thermoelasticity and diffusion has been proposed by Biot [12], [13]. It has the form of a restricted principle wherein the quantity to be varied is the heat displacement vector $\boldsymbol{H}$, defined as the time integral of the heat flux vector $\boldsymbol{J}$. For heat conduction, Biot's criterion may be written as follows:

$$\int \left(\frac{\partial T}{\partial x_i} + \frac{1}{k} \dot{H}_i \right) \delta H_i dV = 0 \tag{1.3}$$

with the law of conservation of energy

$$cT = - \frac{\partial H_i}{\partial x_i} \tag{1.4}$$

introduced as a holonomic constraint. Expressing the heat displacement in terms of generalized coordinates q_α and substituting in (1.3) leads to an equation of Lagrangian form

$$\frac{\partial V}{\partial q_\alpha} + \frac{\partial D}{\partial \dot{q}_\alpha} = Q_\alpha \qquad \alpha = 1, \ldots, n\,, \tag{1.5}$$

which may be easier to solve than the original heat equation (1.1). The quantities V, D and Q, are respectively the thermal potential, the dissipation function and the generalized thermal force; they are well defined functions of T and $\boldsymbol{H}$. Several heat conduction problems have been solved using this method, see for instance ref. [14] and [15].

More recently, Vujanovic and his collaborators [16], [17] presented another variational formulation. Because this theory has given raise to very much interest in the latest years, we shall analyse it in more detail.

There exists no variational principle for the heat conduction equation (1.1). Moreover, this expression, which is parabolic, implies that a thermal disturbance must propagate with an infinite speed. To avoid this unpleasant feature, (1.1) has often [18], [19] been replaced by the hyperbolic equation

$$c\tau \ddot{T} + c\dot{T} = k \frac{\partial^2 T}{\partial x_i^2}\,, \qquad (c, k = \text{constants})\,, \tag{1.6}$$

where τ is the relaxation time.

It has been shown by Vujanovic that there exists an exact variational principle for equation (1.6); this criterion is given by

$$\delta \int_t \int_v \left[\frac{\tau}{2} (\dot{T})^2 - \frac{1}{2} \frac{k}{c} \left(\frac{\partial T}{\partial x_i} \right)^2 \right] e^{t/\tau} dt dV = 0 \qquad (c, k = \text{constants})\,. \tag{1.7}$$

The corresponding Lagrange equation is nothing but (1.6) multiplied by the factor $e^{t/\tau}$. Now, in order to recover equation (1.1) by using (1.7), Vujanovic observes the following rules: at the end of the variational procedure, he drops the factor $e^{t/\tau}$ and afterwards, he takes the limit $\tau \to 0$.

Vujanovic extended his theory to non-linear heat conduction [20] and to isothermal boundary layer flows [21], [22]. This was done by introducing an alias function $\psi(\boldsymbol{x}, t, \lambda)$ depending on an arbitrary parameter λ and obeying the conditions:

$$\lim_{\lambda \to 0} \psi(\boldsymbol{x}, t, \lambda) = 0\,, \quad \lim_{\lambda \to 0} \frac{\partial \psi(\boldsymbol{x}, t, \lambda)}{\partial x} = 1\,, \quad \lim_{\lambda \to 0} \frac{\partial \psi(\boldsymbol{x}, t, \lambda)}{\partial t} = 1\,. \tag{1.8}$$

In our opinion, this theory presents, however, some important shortcomings.

1. The whole procedure rests on the property that equation (1.6) reduces to (1.1) when $\tau \to 0$. This is certainly not true a priori because the solution of (1.6) depends explicitly on τ. Therefore, for each problem, one has to make certain that (1.6) reduces effectively to (1.1) either by solving (1.6) or by using a method like the singular perturbation technique.
2. It is not sure that the function ψ defined by (1.7) exists; no explicit form of this quantity is given in Vujanovic's papers. If it is assumed that ψ is continuously differentiable, it is clear that it does not exist; indeed, a function cannot be simultaneously equal to zero and grow with time and space.

3. When the heat flux is prescribed, or when radiation boundary conditions are applied, surface terms are missing in the expressions of Vujanovic's action integral. This is important because these additional terms must be used in the approximation direct methods.
4. The theory has only been formulated for isotropic bodies. The extension to include anisotropy is not straightforward.
5. Some variational techniques, like the Rayleigh–Ritz method are not applicable because at the limit $\lambda \to 0$, it would yield infinite results. Only the partial integration method can be employed.

Many other variational criteria have been proposed, each of them covering a particular field of continuum mechanics. Among them, let us mention Rosen's principle [23] for heat conduction, Herivel's [24] for ideal fluids flow and Rayleigh–Helmholtz's [25] for isothermal viscous fluids in slow motion.

Recently, one of the authors (G. L.) in collaboration with J. Lambermont presented a general variational principle for purely dissipative processes [26] as well as for fluids in motion [27]. The purpose of this paper is to apply this principle to some non-linear heat conduction problems.

In section 2, the expression of Lebon–Lambermont's criterion for heat conduction is derived. As an illustration, the temperature distribution in a semi-infinite body with thermal properties varying with the temperature is calculated. This is done by using the partial integration method. Two kinds of boundary conditions are considered: in section 3, the surface temperature is prescribed; in section 4, the heat flux across the surface is imposed.

2. Lebon-Lambermont's variational principle

Before deriving the variational criterion, let us briefly recall some definitions. The internal energy of the body is, neglecting deformations, a function of the entropy only. Referring all the quantities to a unit volume, one has

$$u_v = u_v(s_v)\,.$$

The Legendre transformation of u_v with respect to the temperature is defined by

$$u_v - Ts_v \equiv f_v \tag{2.1}$$

and is identical with the Helmholtz free energy. Its time derivative is

$$\dot{f}_v = -s_v \dot{T}\,. \tag{2.2}$$

The entropy production per unit volume due to heat conduction is given by

$$\sigma = J_i \frac{\partial T^{-1}}{\partial x_i}\,, \tag{2.3}$$

where J_i is the ith component of the heat flux vector $\boldsymbol{J}$. Introducing the energy dissipation function

$$\psi = \tfrac{1}{2} T\sigma\,, \tag{2.4}$$

one has, according to (2.3):

$$\psi = -\tfrac{1}{2} \frac{J_i}{T} \frac{\partial T}{\partial x_i}\,. \tag{2.5}$$

For an anisotropic medium, (2.5) suggests the following phenomenological relation

$$J_i = -\frac{L_{ij}}{T} \frac{\partial T}{\partial x_j}\,. \tag{2.6}$$

The phenomenological coefficient L_{ij} is connected with the heat conductivity coefficient by

$$L_{ij} = Tk_{ij}\,, \tag{2.7}$$

where, in virtue of Onsager's reciprocal relations

$$L_{ij} = L_{ji}\,. \tag{2.8}$$

Substitution of (2.6) in (2.5) yields

$$\psi = \tfrac{1}{2}\frac{L_{ij}}{T^2}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}\,. \tag{2.9}$$

If the temperature is prescribed at the surface, the principle of Lebon–Lambermont is expressed by

$$\delta_t I \equiv \delta_t \int_t \int_v L\,dt\,dV = 0\,, \tag{2.10}$$

where the Lagrangian L is given by

$$L = \dot{f}_v - \psi\,. \tag{2.11}$$

The variation is taken with respect to the temperature T, the subscript t means that the time derivative of T must be kept fixed during variation. If L_{ij} is depending on the temperature, it must also be frozen. Using (2.2), (2.7) and (2.9), L may be written as:

$$L = -s_v \dot{T}^* - \frac{1}{2T^2}k^*_{ij}T^*\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j} \equiv L\left(T, \frac{\partial T}{\partial x_i}, t, x_i\right). \tag{2.12}$$

An asterisk reminds one that the corresponding quantities are to be held constant during the variational procedure.

The Euler–Lagrange equation associated with (2.10) is

$$\frac{\partial L}{\partial T} - \frac{\partial}{\partial x_i}\frac{\partial L}{\partial\left(\frac{\partial T}{\partial x_i}\right)} = 0$$

i.e.

$$-\frac{\partial s_v}{\partial T}\dot{T}^* + k^*_{ij}\frac{T^*}{T^3}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j} + \frac{\partial}{\partial x_i}\left\{\frac{T^*}{2T^2}(k^*_{ij}+k^*_{ji})\frac{\partial T}{\partial x_j}\right\} = 0\,. \tag{2.14}$$

At this stage of the calculations, the asterisks must be dropped. Making use of the Onsager relations and of the classical result

$$ds_v = c\,\frac{dT}{T}\,, \tag{2.15}$$

(2.14) reduces to

$$-c\frac{\dot{T}}{T} + \frac{1}{T}\frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial T}{\partial x_j}\right) = 0 \tag{2.16}$$

which is the equation for heat conduction in an anisotropic solid.

Instead of imposing the temperature, consider that the heat flux is a prescribed function $g(x, t)$ of space and time over a part A_J of the surface A and that a radiation type condition like

$$\boldsymbol{J}\cdot\boldsymbol{n} = h(T - T_0)$$

is applied over the part A_r of $A = A_J \cup A_r$; h is the heat transfer coefficient and T_0 the temperature of the surrounding medium.

The functional I must then be modified as follows

$$I = \int_t \int_v (\dot{f}_v - \psi)\,dV\,dt - \int_t \int_{A_J} \frac{g_i}{T^*} n_i T\,dA\,dt - \tfrac{1}{2}\int_t \int_{A_r} \frac{h(T-T_0)^2}{T^*}\,dA\,dt\,, \tag{2.18}$$

g_i is the ith component of the prescribed vector $\boldsymbol{g}$, n_i is the ith component of the unit normal pointing outwards. The conditions for I to be stationary are now the energy law (1.1) plus the boundary conditions

$$-k_{ij}\frac{\partial T}{\partial x_j}n_i = g_i n_i \qquad \text{on } A_J\,, \tag{2.19}$$

$$-k_{ij}\frac{\partial T}{\partial x_j}n_i = h(T-T_0) \qquad \text{on } A_r\,. \tag{2.20}$$

3. Heat conduction in a semi-infinite solid with temperature prescribed at the boundary

Consider an isotropic, homogeneous, semi-infinite solid bounded by the plane $x=0$ and extending to infinity in the direction of positive x. The solid is initially at the uniform temperature T_0.

At $t=0$, the face $x=0$ is suddenly brought to the temperature $2T_0$:

$$\begin{aligned} &T=T_0 \quad \text{for} \quad t=0 \quad \text{and} \quad \forall x\,,\\ &T=2T_0 \quad \text{for} \quad \forall t \quad \text{and} \quad x=0\,. \end{aligned} \tag{3.1}$$

The thermal conductivity and the heat capacity are supposed to be of the form

$$k=k_0(1+\alpha\theta)\,, \tag{3.2}$$

$$c=c_0(1+\beta\theta)\,, \tag{3.3}$$

with

$$\theta = T/T_0\;; \tag{3.4}$$

k_0, c_0, α, β are given constants.

With the above expressions for k and c, the action integral I is given by

$$I = \int_t\int_x\left\{-c_0(\beta\theta+\ln\theta)\dot{\theta}^* - \tfrac{1}{2}\,\frac{k_0(1+\alpha\theta^*)\theta^*}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2\right\}dx\,dt\,. \tag{3.5}$$

In order to obtain an approximate solution for the temperature distribution, we use the method of partial integration.

Following Biot, we assume that the temperature distribution is parabolic:

$$\begin{aligned} &\theta(x,t) = 1+\left(1-\frac{x}{f(t)}\right)^2 \qquad x \leqq f(t)\,,\\ &\theta(x,t)=1 \qquad\qquad\qquad\quad\; x \geqq f(t)\,, \end{aligned} \tag{3.6}$$

$f(t)$ is an unknown function of time and may be interpreted physically as being the penetration depth. According to (3.1), one has

$$f(0)=0\,. \tag{3.7}$$

Substitute (3.6) in (3.5) and integrate with respect to x, the limits of integration being $x=0$ and $x=f$, one gets

$$\begin{aligned} I = 2\int_0^t\Bigg[&\frac{\dot{f}^*}{f^{*3}}f^2\left(\beta\left(\frac{7}{12}f^*-\frac{11}{30}f\right)+\left(\frac{\pi}{2}-\frac{3}{2}\right)f^*+\left(\frac{2}{3}\ln 2-\frac{\pi}{3}+\frac{5}{9}\right)f\right)\\ &+\frac{\kappa_0}{f^{*4}}\left\{\frac{1}{2}\frac{f^{*4}}{f}\left(\frac{\pi}{2}-1\right)(1+2\alpha)-ff^*\left(\left(\frac{\pi}{4}-\frac{3}{2}\right)(1+8\alpha)+\ln 2\,(1-8\alpha)\right).\right.\\ &-f^{*3}\left(\frac{\pi}{4}(1+4\alpha)-\ln 2\,(1-4\alpha)\right)+f^3\alpha\left(6\ln 2-\frac{3\pi}{2}+\frac{10}{3}\right)\\ &\left.+\,2\alpha f^2 f^*(2\pi-7-5\ln 2)\right\}\Bigg]dt \equiv \int_0^t L(f,t)\,dt \end{aligned} \tag{3.8}$$

where $\kappa_0 = k_0/c_0$ is the diffusivity.

The Euler–Lagrange relation corresponding to the variational equation $\delta I = 0$ is given by

$$\frac{\partial L}{\partial f} = 0 . \tag{3.9}$$

It must be recalled that, while derivating L with respect to f, all the quantities with an asterisk are to be held fixed. When all the derivations are performed, the asterisks may be dropped and (3.9) yields:

$$f\dot{f}(0.0537+0.0666\,\beta) = \kappa\,(0.2643+0.4721\,\alpha) . \tag{3.10}$$

This is a first order differential equation with respect to the time. With the initial condition (3.1), one obtains

$$f = \gamma\,[\kappa_0 t]^{\frac{1}{2}} \tag{3.11}$$

with

$$\gamma = \left(\frac{2(0.2643+0.4721\,\alpha)}{0.0537+0.0666\,\beta}\right)^{\frac{1}{2}} .$$

The values of γ corresponding to different values of k and c are reported in table 1.

TABLE 1

Values of γ

c \ k	k_0	$k_0(1+\frac{1}{2}\theta)$	$k_0(1+\theta)$	$k_0(1-\frac{1}{2}\theta)$
c_0	3.14	4.32	5.24	1.03
$c_0(1+\theta)$	2.10	2.88	3.50	0.69

The results obtained in the theory of Biot are also of the form (3.11). However, to compare our results with those of Biot, one has to assume that the initial dimensionless temperature is uniformly zero instead of one. This implies that in the expressions (3.2) to (3.6), θ must be replaced by $1+\bar{\theta}$ (the upper bar indicates that the initial temperature has been chosen equal to zero).

The corresponding values of γ are given in table 2 and compared with Biot's ones.

TABLE 2

Values of γ

c \ k	k_0	$k_0(1+\frac{1}{2}\bar{\theta})$	$k_0(1-\frac{1}{2}\bar{\theta})$
c_0	3.14 3.36 (Biot)	3.70 3.83 (Biot)	2.434 2.76 (Biot)
$c_0(1+\bar{\theta})$	2.82 2.97 (Biot)		

In fig. 1, we have represented the approximate value of $\bar{\theta}$ versus the quantity $\xi = x/2(\kappa_0 t)^{\frac{1}{2}}$ for different values of k and c. We have also plotted the corresponding results obtained by means of Biot's method; the exact solution for c and k constant has also been represented. It may be seen that our results differ slightly from Biot's and fit rather well the exact solution.

For metals, the heat conductivity law (3.2), expressing k as a linear function of θ is not very satisfactory. A more appropriate dependence is [28]:

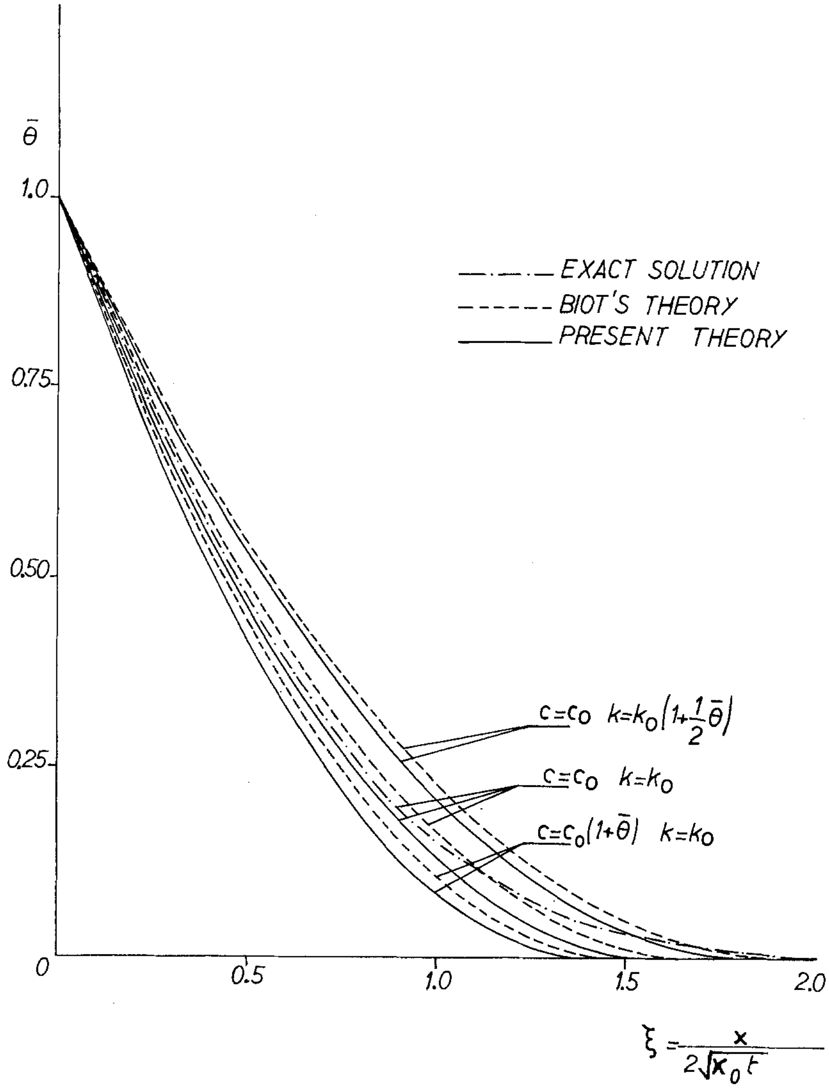

Figure 1. Internal temperature profile in a semi-infinite body with prescribed temperature at the boundary.

$$k = \frac{k_0}{\theta} \quad \text{(Eucken's law)}. \tag{3.12}$$

With this form for k, the functional I is given by:

$$I = \int_t \int_x \left\{ -(\beta\theta + \ln \theta)\dot{\theta}^* - \tfrac{1}{2}\frac{\kappa_0}{\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 \right\} dx\,dt \tag{3.13}$$

wherein $\dot{\theta}$ is the only quantity to be kept fixed.

Substitute the trial function (3.6) in (3.13) and integrate with respect to the space coordinate. The corresponding Euler–Lagrange equation yields

$$f\dot{f} = \kappa_0 \frac{0.1427}{0.0537 + 0.0666\,\beta}. \tag{3.14}$$

The values of f for $\beta=0$ (constant heat capacity) and $\beta=1$ are respectively

$$f = 2.31\,[\kappa_0 t]^{\frac{1}{2}} \qquad (\beta = 0)$$
$$f = 1.54\,[\kappa_0 t]^{\frac{1}{2}} \qquad (\beta = 1)$$

By using Biot's method, one would find in the case $\beta=0$,

$$f = 2.64\,[\kappa_0 t]^{\frac{1}{2}}\,.$$

The above analysis may be easily extended to bodies with non-linear thermal properties. It would only result in more complicated algebraic operations.

4. Heat conduction in a semi-infinite body with prescribed heat flux at the surface

The semi-infinite solid described in section 3 is heated at the surface $x=0$ by a heat flux which is a given function of the time:

$$J \equiv -k\frac{\partial\theta}{\partial x} = g(t)^{\star} \qquad \text{for} \quad x=0\,. \tag{4.1}$$

The heat conductivity and the heat capacity are taken to be given by (3.2) and (3.3) and the initial temperature is taken equal to zero:

$$\theta(x,0) = 0 \quad \forall x\,. \tag{4.2}$$

In order to take the boundary condition (4.1) into account, the variational method of finding an approximate solution must be adapted. Here we employ the technique proposed by Lardner [14] and Rafalski and Zyszkowski [29].

We choose as trial function

$$\theta = q\,\mathrm{e}^{-(fx+1)^2} \tag{4.3}$$

wherein, according to the above mentioned procedure, two parameters f and q are introduced. Both are unknown functions of the time but $f(t)$ is the independent parameter to be calculated by means of the variational method while $q(t)$ is treated as a given function not subject to variations; $q(t)$ will be determined by the boundary condition (4.1) which is no part of the variational process.

Note that $q(t)$ is related with the temperature at the surface by

$$q(t) = \mathrm{e}\,\theta(0,t)\,, \tag{4.4}$$

with, according to (4.2),

$$q(0) = 0\,. \tag{4.5}$$

Inserting (4.3) in the expression (2.18) of the action integral results in

$$\begin{aligned} I = &-\int_t\int_0^\infty \Big\{ \mathrm{e}^{-(f^*x+1)^2}[\dot{q} - 2qx(f^*x+1)\dot{f}^*][\beta q\,\mathrm{e}^{-(fx+1)^2} - (fx+1)^2 + \ln q] \\ &+ 2\frac{k_0}{c_0}\,qf^2(fx+1)^2\,\mathrm{e}^{-(f^*x+1)^2}(1+\alpha q\,\mathrm{e}^{-(f^*x+1)^2})\Big\}\,dx\,dt \\ &-\int_t |g\,\mathrm{e}^{-(fx+1)^2+(f^*x+1)^2}|_{x=0}\,dt\,. \end{aligned} \tag{4.6}$$

After integration with respect to the time, the above functional may be written in the form:

$$I = \int_t L(f,t)\,dt\,. \tag{4.7}$$

The Euler–Lagrange equation, ensuring that I is stationary, is expressed by

$$\frac{\partial L}{\partial f} = 0\,. \tag{4.8}$$

After performing the derivation with respect to f, we identify f^* and $\dot{f}^*$ with f and $\dot{f}$ respectively so that (4.8) gives:

* For convenience, the non-dimensional temperature will be used.

$$\frac{k_0}{c_0}\, qf^2 \left\{2\left(\frac{1}{e} + \sqrt{\pi}\, \operatorname{erfc} 1\right) + \alpha q\left[\frac{1}{e^2} + \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \operatorname{erfc} \sqrt{2}\right]\right\}$$

$$-\dot{q}\,\frac{\pi^{\frac{1}{2}}}{2} \operatorname{erfc} 1 + q\,\frac{\dot{f}}{f}\left(5\,\frac{\pi^{\frac{1}{2}}}{2} \operatorname{erfc} 1 - \frac{1}{e}\right) - q\beta\left[\tfrac{1}{4}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \dot{q}\, \operatorname{erfc} \sqrt{2}\right.$$

$$\left. -q\,\frac{\dot{f}}{f}\left(\tfrac{7}{8}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \operatorname{erfc} \sqrt{2} - \frac{1}{4e^2}\right)\right] = 0\,, \tag{4.9}$$

where erfc y is the complementary error function

$$\operatorname{erfc} y = \frac{2}{\pi^{\frac{1}{2}}} \int_y^{\infty} e^{-\zeta^2} d\zeta\,. \tag{4.10}$$

By introducing the expression of the trial function (4.3) in the boundary law (4.1), one obtains the algebraic equation

$$k_0\left(1 + \frac{\alpha q}{e}\right)\frac{2qf}{e} = g(t)\,, \tag{4.11}$$

which, combined with (4.9) constitutes a set of two equations for the two unknowns q and f.

Let us first consider *solids with constant thermal properties* ($\alpha = 0$, $\beta = 0$); (4.11) reduces then to

$$f = \frac{eg(t)}{2k_0 q}\,. \tag{4.12}$$

Elimination of f between (4.9) and (4.12) results in the first order differential equation

$$q\dot{q} - 0.7024\, q^2\, \frac{\dot{g}}{g} = 0.6901\, \frac{e^2 g^2}{k_0 c_0}\,. \tag{4.13}$$

If the heat flux is constant through the boundary ($\dot{g} = 0$), the solution of (4.13) is

$$q = 1.17\, ge\left(\frac{t}{k_0 c_0}\right)^{\frac{1}{2}}. \tag{4.14}$$

By dividing this result by e, one obtains the expression of the surface temperature, Lardner [14] has treated the same problem in the frame of Biot's theory; taking a parabolic profile for the spatial temperature distribution he gets

$$\theta(0, t) = 1.12\, g\left(\frac{t}{k_0 c_0}\right)^{\frac{1}{2}}. \tag{4.15}$$

Choosing a cubic profile, Vujanovic [17] finds

$$\theta(0, t) = 1.128\, g\left(\frac{t}{k_0 c_0}\right)^{\frac{1}{2}}, \tag{4.16}$$

while by applying the integral method and a parabolic profile, Goodman [30] obtains

$$\theta(0, t) = 1.225\, g\left(\frac{t}{k_0 c_0}\right)^{\frac{1}{2}}. \tag{4.17}$$

In all cases, the agreement with the exact solution [1]

$$\theta(0, t) = 1.128\, g\left(\frac{t}{k_0 c_0}\right)^{\frac{1}{2}},$$

is seen to be quite satisfactory.

If it is assumed that g follows a power law of the form

$$g = t^{n/2} \qquad (n \geqq 0)\,, \tag{4.20}$$

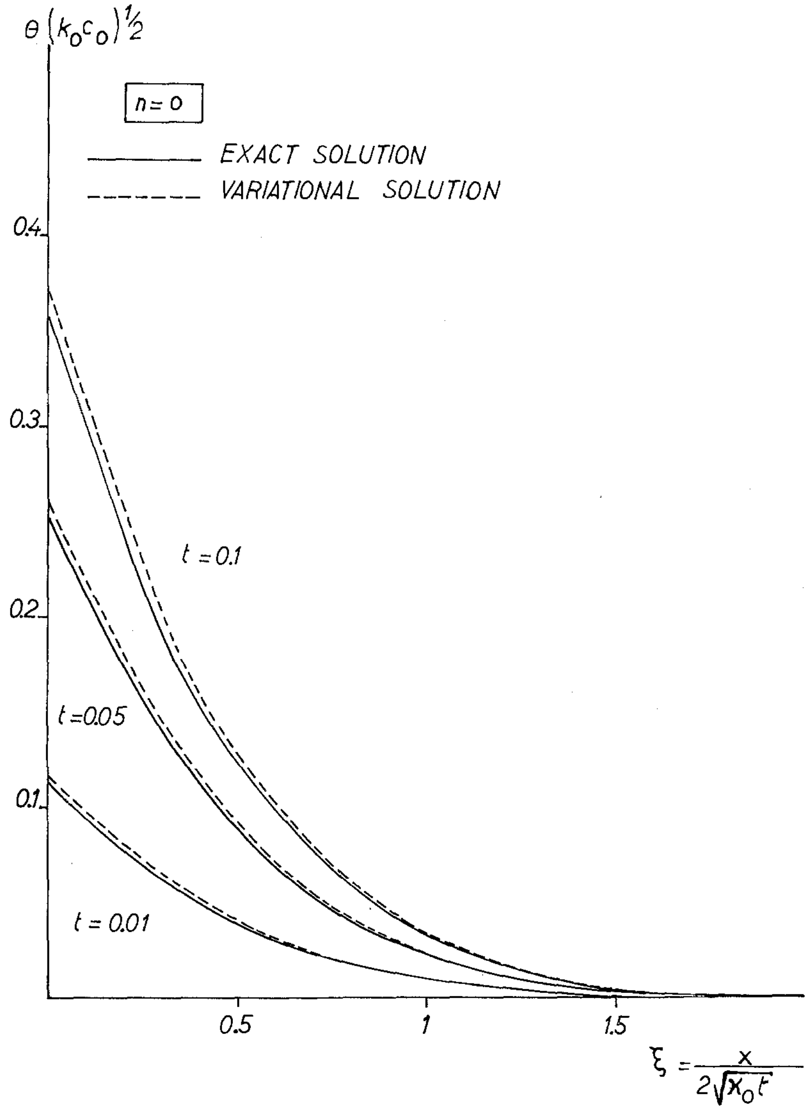

Figure 2. Internal temperature distribution in a semi-infinite body with constant heat flux prescribed at the boundary.

expression (4.13) becomes

$$\frac{d}{dt}\left(\tfrac{1}{2}q^2\right) - 0{,}7024\,\frac{n}{2t}\,q^2 = \frac{e^2}{k_0 c_0}\,0{,}6901\,. \tag{4.21}$$

The solution of (4.21) is

$$q = \frac{eA}{(k_0 c_0)^{\frac{1}{2}}}\,t^{\frac{1}{2}(n+1)} \tag{4.22}$$

where

$$A = \left(\frac{1{,}38}{1 + 0{,}5959\,\dfrac{n}{2}}\right)^{\frac{1}{2}}.$$

Introducing (4.22) in (4.3) and setting

$$\xi = \frac{x}{2(\kappa_0 t)^{\frac{1}{2}}}\,,$$

one obtains the temperature profile inside the body:

$$\theta = \frac{eA}{(k_0 c_0)^{\frac{1}{2}}} t^{\frac{1}{2}(n+1)} \exp -\left(\frac{\xi}{A} + 1\right)^2 . \tag{4.23}$$

The exact solution is known to be

$$\theta = \frac{\Gamma(\frac{1}{2}n+1)}{(k_0 c_0)^{\frac{1}{2}}} (4t)^{\frac{1}{2}(n+1)} i^{n+1} \operatorname{erfc} \xi , \tag{4.24}$$

where $i^n \operatorname{erfc} \xi$ is defined by [1]

$$i^n \operatorname{erfc} \xi = \int_x^\infty i^{n-1} \operatorname{erfc} \xi \, d\xi .$$

The approximate and exact solutions are compared in fig. 2 and 3 for $n=0$ and $n=1$ respectively. It is seen that the choice of the trial function (4.3) is less appropriate for $n=1$ than for $n=0$.

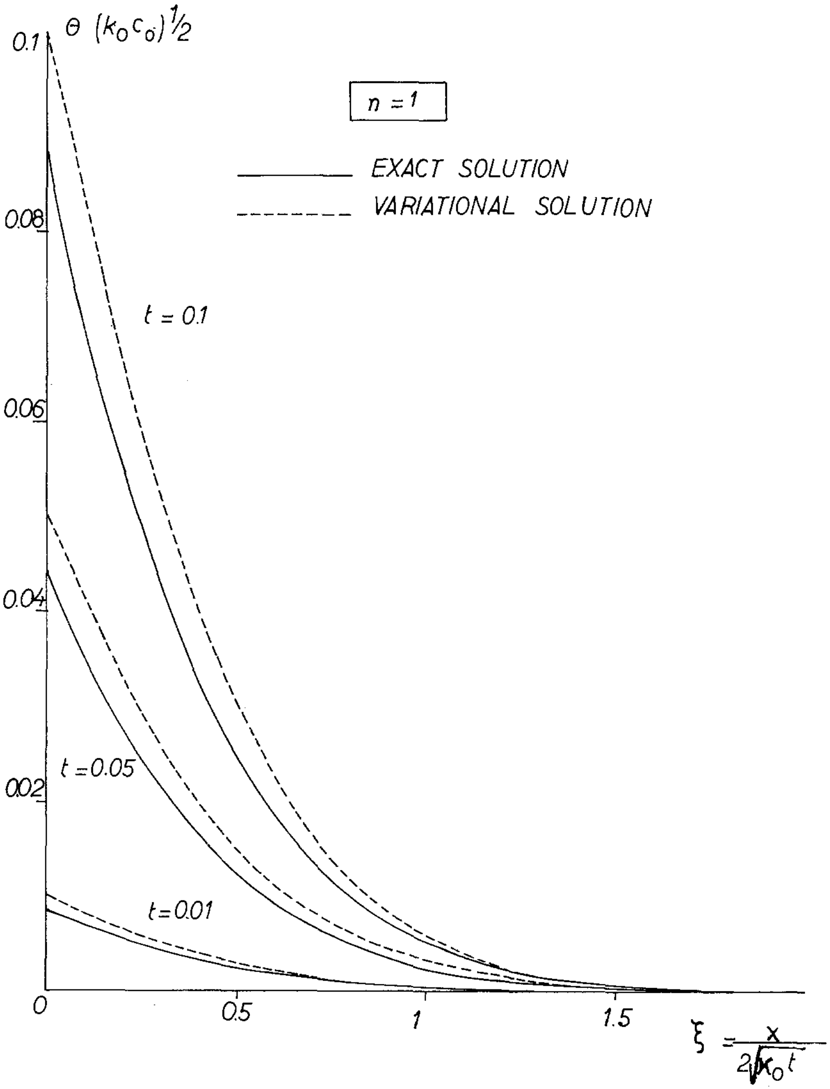

Figure 3. Internal temperature distribution in a semi-infinite body with heat flux at the boundary proportional to the square root of the time.

Consider now a solid with *constant conductivity but variable heat capacity* of the form (3.2). If the heat flux is supposed constant, substitution of (4.12) in (4.9) leads to:

$$q\dot{q}+0.6444\,\beta q^2\dot{q}=0.6901\,\frac{e^2g^2}{k_0c_0}\,. \tag{4.25}$$

After integration and setting

$$Q=q/e\equiv\theta(0,t)\,, \tag{4.26}$$

one gets

$$\frac{Q^2}{2}+0.06444\,\beta\,\frac{Q^3}{3e}=0.6901\,\frac{g^2}{k_0c_0}\,t\,. \tag{4.27}$$

The values of Q are plotted on fig. 4 in terms of τ $(=g^2t/k_0c_0)$ for $\beta=1, 0.5, 0, -0.5$ and -1 respectively.

Finally we assume that the *specific heat is constant but that the heat conductivity varies* linearly with the temperature. For a constant g, elimination of f between (4.9) and (4.12) yields

$$0.6901\,\frac{g^2}{k_0c_0}=Q\dot{Q}(1+\alpha Q)(1+1.7025\,\alpha Q)/(1+0.4039\,\alpha Q)\,. \tag{4.28}$$

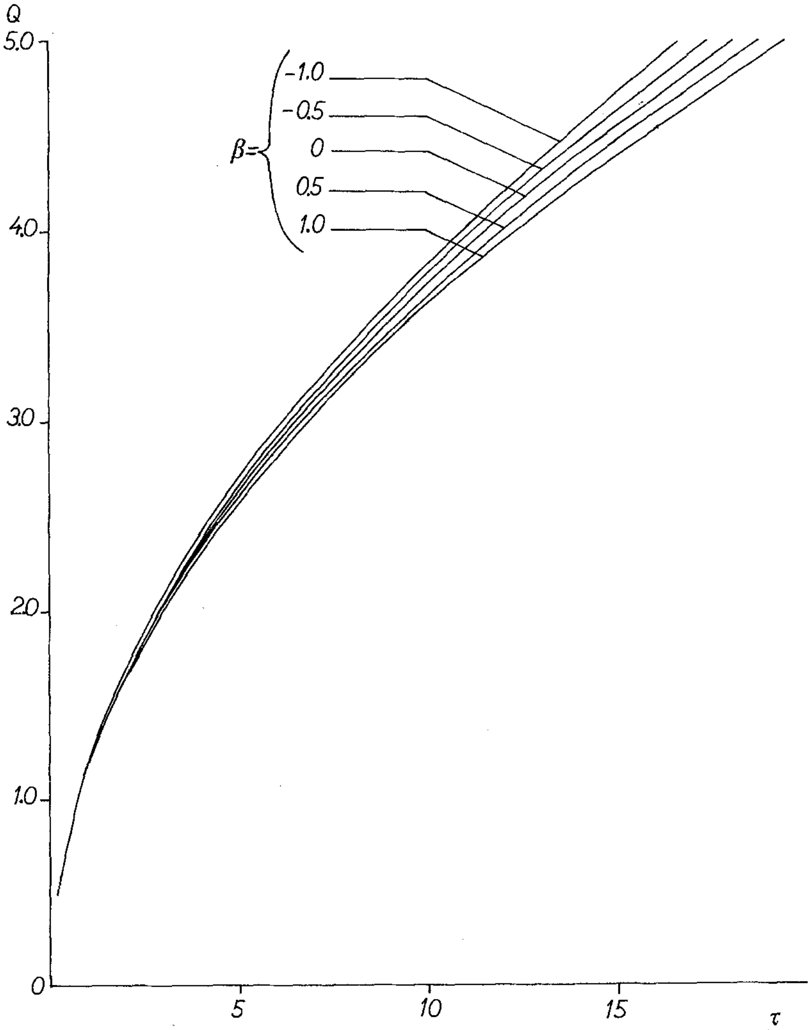

Figure 4. Effect on varying the heat capacity on the surface temperature history

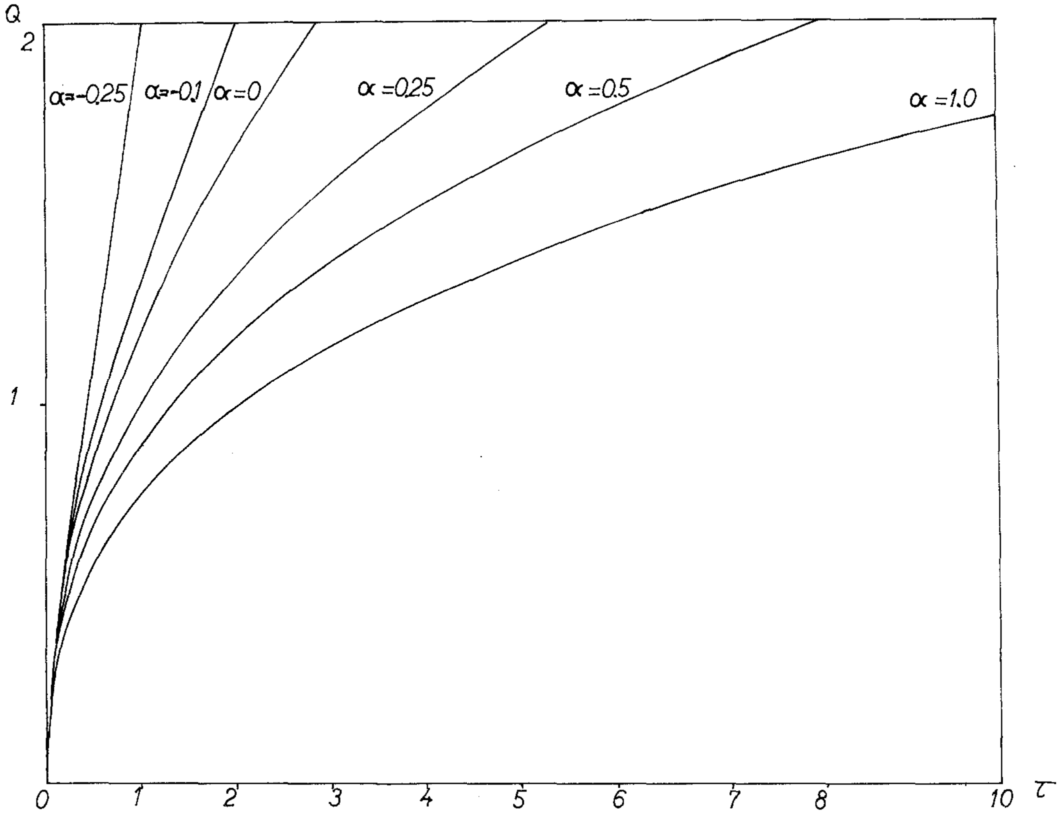

Figure 5. Effect on varying the heat conductivity on the surface temperature history.

The solutions have been calculated for $\alpha = -0.25, -0.1, 0, 0.25, 0.5, 1$ and are represented by fig. 5.

5. Concluding remarks

In all the cases where analytic results are available, it has been seen that the variational solution constitutes a good approximation of the true one. This justifies the use of variational methods in treating complicated problems, like those involving temperature-dependent thermal properties, which cannot be solved directly.

The choice of the selected variational principle is, to a certain extent, of sentimental nature. Besides, several principles may be used together to test the accuracy of the solution obtained. Indeed, if two or more principles give approximately the same result, it is reasonable to think that it is reliable. However, for certain classes of problems, some principles lead to heavier numerical calculations than others. It must also be kept in mind that some formulations like those of Biot and Vujanovic, are less general than those of Glansdorff–Prigogine and Lambermont–Lebon. The former are more adapted to describe heat transfer and related situations, the latter embrace a larger field of macroscopic physics including chemical reactions, diffusion processes and fluid flows. Moreover, the very construction of Biot's and Vujanovic's criteria imply that the partial integration method must be employed to the exclusion of any other technique like Rayleigh–Ritz's or the iteration method.

In this paper, we have chosen very simple trial functions involving at the maximum two parameters. Moreover, all the computations have been performed with the help of a minipocket computer. The values of the quantities α and β were chosen for illustrative purpose and are not necessarily representative of a particular material; the object of this paper was not to solve a given practical problem but rather to show that our method may be of great use in solving non-linear heat transfer problems.

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